A Continuous Time Model for Correlated Energy Price Processes

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Abstract

A continuous time model of two correlated mean reverting processes is proposed to describe important issues in integrated energy risk management: hedging in energy production and optimal timing of technical investments. Since in both problems the decision maker faces uncertainty both in the price of input (typically oil, gas or other commodity necessary for production) and in the price of output (electricity), we show that the positive correlation between input and output prices can be exploited to help reducing the production risk. First we consider the problem of a small energy producer willing to optimize his sales by signing a forward and/or selling spot, secondly we face the problem of the search of the optimum time in which invest in an energy plant. The theoretical results are accompanied by empirical applications from the german EEX market.

1. Introduction

Energy markets are strongly integrated and volatile. A major reason for the high volatility is that global and local energy demand is highly inelastic in the short time and any shortage in the availability of one resource generates an immediate and large rise in the demand of the other resources, directly or indirectly related, thereby affecting prices. On the other hand economic growth, environmental
changes and progressive exhaustion of non-renewable resources make it difficult to put in place, at least in the mid-horizon, plants and systems to guarantee local and global over-capacity which should be required to keep the volatility controlled. So shocks from both demand and supply side of energy resources are expected to significantly affect the energy markets in the short-medium range. Mathematical models have appeared in the literature to describe the stochastic behaviour of energy prices ([1]; [2]; [3]; [10]; [11]; [12]; [14]; [17]). Those models suffer from a major drawback as they miss the multidimensional nature of energy markets. Intrinsic difficulties in the thorough design of even a single process are a major technical obstacle, especially when referring to electricity prices, where one has to introduce mathematical devices to capture the random presence of spikes. In this framework see also the work of [5], where a multidimensional process is illustrated (coupling prices of fuel and electricity), assuming that the Brownian motions are correlated. Nevertheless, the need to model energy price processes under a unified multidimensional model is of major importance in optimal decisions to take, both at a corporate and at a market level. As an example we quote two typical problems in electricity production: short period risk management and long period technical investments, which can not be satisfactorily obtained by means of simulations of a single process. Referring specifically to the mentioned examples, we will try a new approach to model both of them, proposing a unified approach based on two correlated mean reverting processes. We refer in this matter to integrated risk management in energy markets [9], even though we claim that our model can be easily extended to other economic sectors. In [6] an analysis is proposed of a large number of risk sources faced both by financial and non financial firms. In this paper we assume that our decision maker is concerned with both the mean and the variance of profits (in this we follow [4]). In managing the integrated risk, we consider ”upstream” (natural gas and fuel) and ”downstream” (electricity) commodities. Since fuel can be converted into electricity, they are strongly related [15] and we expect a positive correlation between them. This correlation can be exploited successfully to reduce the risk of a producer who has to take a decision of selling in the spot and/or forward market or has to decide the optimal timing in plant investment. We describe fuel and electricity prices as governed by two correlated exponential mean reverting processes. A linear correlation coefficient allows to capture the common dependency of the two processes from common risk factors. Analytical expressions of the relevant decision parameters are obtained and estimates based on maximum likelihood are provided. The first problem we face here regards an energy producer participating to the spot market and acting
as a price taker. The producer faces two alternatives: selling the production in the
spot market and/or signing a forward contract. Such a decision is not obvious in
a risk return setting, since the producer can easily take advantage of the positive
correlation between input and output prices to reduce the risk exposure. Relying
on a pure forward selling introduces a rigidity in the profit function, since the
selling price becomes a constant but producing costs remain free to fluctuate (we
assume that there are no forwards on the fuel) and diversification opportunities are
lost. We show that an efficient frontier exists of optimal spot/forward allocation;
in order to further help the decision maker we introduce the Sharpe Ratio (a
very common tool in finance) and identify the optimal allocation maximizing the
Sharpe Ratio. For a thorough description of the general model see [7]. Simulations
performed for various values of the involved parameters show that the maturity
of the forward plays an important role in the optimal choice. An application to
the EEX german market is developed.

The second problem we approach is related to a timing decision of investment,
I.e. which is the optimal time to invest in an energy production plant. The
decision problem is dependent on two state variables which are the correlated
mean reverting processes governing the price of the input and output. We show
that correlation is again a relevant factor for both the corporate decisions problems
analysed here.

The paper is organized as follows: in Section 1 we describe the general prop-
erties of an exponential mean reverting process. The extension to a bidimensional
model is illustrated in Section 2, that provides also the parameters estimation.
Risk management is in Section 3. Optimal timing is in Section 4. In Section 5
both problems are applied to the German market.

Conclusions are in the last Section.

2. Exponential mean reverting processes

An exponential mean reverting process is the exponential of a standard mean
reverting process. Similarly to a geometric brownian motion, the random variable
is positive, which is a necessary property to describe prices. An exponential mean
reverting process is the solution of the stochastic equation (1):

\[ dz = -\lambda z (\log(z) - f) \, dt + \sigma z \, dW \]  \hspace{1cm} (1)

Then \( z = e^x \), where \( x \) is the mean reverting process which solves equation (2)
\[ dx = -\lambda (x-f) \, dt + \sigma \, dW \]  \hspace{1cm} (2)

If \( z \) is a price, \( dx = d(\log (z)) \) can be interpreted as the instantaneous growth rate since

\[ z(t+\delta t) = e^{x+\delta x} = e^x e^{\delta x} \simeq e^x (1 + \delta x) = z + \delta x \, z \]

Modelling the rate by a mean reverting process is like assuming that there is a benchmark \( m \) that calls the rate back as \( x \) deviates from it. The parameter \( f \) can be time dependent; when it is a constant it is called long run mean since it is the limit of the expected value of \( x(t) \) as \( t \) tends to infinity. This value is independent on \( x(0) \), this is the reason why it is actually a benchmark. When \( f = f(t) \) it just represents the benchmark value for \( x(t) \): even if \( x(0) = f(0) \), neither \( E[x(t)] = f(t) \) nor (usually) \( |E[x(t)] - f(t)| \overset{t \to \infty}{\to} 0 \). Time dependent means are useful to describe seasonalties.

Since Brownian motions belong to the class of mean reverting processes (when \( \lambda = 0 \), \( x(t) \) is a brownian motion without drift), the geometric brownian motion belongs to the class of exponential mean reverting processes. Geometric brownian motions with drift \( \mu \) can be just approximated: taking \( \lambda \sim 0 \) and \( f = \frac{\mu}{\lambda} \).

On the opposite side, as \( \lambda \to \infty \), \( x(t) \) tends to a white noise. Then \( z(t) \) tends to a white noise with jumps log-normally distributed.

In general, \( x(t) \) (given \( x(0) \)) is normally distributed, then \( z(t) \) (given \( z(0) \)) is log-normally distributed: known \( x(0) \), the solution \( x(t) \) of a mean reverting with constant reversion \( \lambda \) and volatility \( \sigma \), but time dependent mean \( f(t) \) is

\[ x(t) \sim N (E[x(t)], \sigma [x(t)]) \]  \hspace{1cm} (3)

\[ E[x(t)] = x(0) e^{-\lambda t} + \lambda \int_{0}^{t} f(\tau) e^{-\lambda (t-\tau)} d\tau \]

\[ \sigma [x(t)] = \sigma \sqrt{\frac{1 - e^{-2\lambda t}}{2\lambda t}} \]

Then \( z(t) \) (given \( z(0) \)) is log-normal with
\begin{equation}
E[z(t)] = e^{E[x(t)] + \frac{\sigma^2[x(t)]]}{2}} \\
\sigma[z(t)] = \sqrt{e^{2\sigma^2[x(t)]} - 1} e^{E[x(t)] + \frac{\sigma^2[x(t)]]}{2}}
\end{equation}

The diffusion coefficient of $z$ is proportional to $z$ itself. This means that $z$ gets "noisy" when $z$ is high. Considering the case when $f$ is a constant: $x(t)$ remains bounded "near" $f$ (the strength of the binding depends on $\lambda$ and $\sigma$). The volatility is asymmetric with respect to $f$: it is higher when $z$ is high and lower when $z$ is low. When the volatility is small, $e^z$ can be linearly approximated around its mean. The exponential of a mean reverting does not significantly differ from a mean reverting in this case (e.g. Fig. 1). But when the volatility is high, high prices are emphasized (e.g. Fig. 2) and the model can generate the spikes\footnote{Actually a MR relaxes towards his mean, and so does its exponential. Usually price spikes fall more abruptly, that is why jump processes have been developed for electricity prices.} which are often observed in the real electricity price series. This is a good reason to adopt the exponential of a mean reverting to describe energy price processes.

![Figure 1: An exponential mean reverting process with low volatility.](image)

The positivity of $z$ and the skewness of its probability distribution is not the only asymmetry of such a process. As an example consider how $E[z(t)]$ evolves when $f$ is a constant: while $E[\ln(z(t))] = E[x(t)]$ relaxes towards $f$ symmetrically and monotonically, the relaxation of $E[z(t)]$ towards $\exp\left(f + \frac{\sigma^2}{2\lambda}\right)$
is asymmetric, besides it is not monotonic if $\exp(f) < z(0) < \exp\left(f + \frac{\sigma^2}{2\lambda}\right)$ (see fig. 3).

We extend this model to a bivariate case, where the dependence on common random factors is represented by a correlation coefficient $\rho$ between stochastic shocks.
2.1. Correlated processes

We consider two exponential mean reverting processes \( p \) and \( c \), which can be thought for example as the selling price of a unit of energy and the corresponding fuel cost required to produce it:

\[
\begin{align*}
    c &= e^x \\
    p &= e^y \\
    dx &= -\lambda_x (x - f_x(t)) \, dt + \sigma_x \, dW_1 \\
    dy &= -\lambda_y (y - f_y(t)) \, dt + \sigma_y \left( \rho \, dW_1 + \sqrt{1 - \rho^2} \, dW_2 \right)
\end{align*}
\]

(5)

where \( \rho = \text{corr}(dx, dy) \).

The functions \( f_x(t) \) and \( f_y(t) \) are supposed known and determine the seasonality of the process. Since \( \rho \) is the correlation between the innovations, it is also the correlation between \( x(t) \) and \( y(t) \), when \( x(0) \) and \( y(0) \) are known. The exponential function weakens the strength of the direct correlation between \( p \) and \( c \). If \( x \) and \( y \) were perfectly correlated, they would have a linear dependence, but \( p \) and \( c \) would not.

The seasonal effects can be removed by a change of variables:

\[
\begin{align*}
    F_x(t) &= x(0) \, e^{-\lambda_x t} + \lambda_x \int_0^t f_x(\tau) \, e^{-\lambda_x (t-\tau)} d\tau \\
    F_y(t) &= y(0) \, e^{-\lambda_y t} + \lambda_y \int_0^t f_y(\tau) \, e^{-\lambda_y (t-\tau)} d\tau \\
    \xi(t) &= x(t) - F_x(t) \\
    \varphi(t) &= y(t) - F_y(t)
\end{align*}
\]

Equations (5) become

\[
\begin{align*}
    c &= e^{F_x(t)} \, e^\xi \\
    p &= e^{F_y(t)} \, e^{\varphi} \\
    d\xi &= -\lambda_x \, \xi \, dt + \sigma_x \, dW_1 \\
    d\varphi &= -\lambda_y \, \varphi \, dt + \sigma_y \left( \rho \, dW_1 + \sqrt{1 - \rho^2} \, dW_2 \right)
\end{align*}
\]

(6)

The solution of the system of stochastic equations (6) has already mentioned and, in particular, the following statistical properties hold
\[
\begin{align*}
\xi(t) &\sim N\left(0, \sigma_\xi = \sigma_x \sqrt{\frac{1-e^{-2\lambda_x t}}{2\lambda_x}}\right) \\
\varphi(t) &\sim N\left(0, \sigma_\varphi = \sigma_y \sqrt{\frac{1-e^{-2\lambda_y t}}{2\lambda_y}}\right)
\end{align*}
\] (7)

The bivariate probability distribution is:

\[
P(\xi, \varphi) \cdot d\xi \cdot d\varphi = \frac{\exp\left(-\frac{1}{2(1-\rho^2)} \left(\frac{\xi^2}{\sigma_\xi^2} + \frac{\varphi^2}{\sigma_\varphi^2} - 2\rho \frac{\xi \varphi}{\sigma_\xi \sigma_\varphi}\right)\right)}{2\pi \sqrt{1-\rho^2} \sigma_\xi \sigma_\varphi} \cdot d\xi \cdot d\varphi \] (8)

From (5) and (7) we obtain that, \(c(0)\) and \(p(0)\) given, \(c(t)\) and \(p(t)\) are log-normal with:

\[
\begin{align*}
\ln c(t) &\sim N\left(F_x(t), \sigma_\xi = \sigma_x \sqrt{\frac{1-e^{-2\lambda_x t}}{2\lambda_x}}\right) \\
\ln p(t) &\sim N\left(F_y(t), \sigma_\varphi = \sigma_y \sqrt{\frac{1-e^{-2\lambda_y t}}{2\lambda_y}}\right)
\end{align*}
\] (9)

The expected values of \(c(t)\) and \(p(t)\) are equal to

\[
\begin{align*}
E[c(t)] &= \exp\left(F_x(t) + \frac{1-e^{-2\lambda_x t}}{4\lambda_x} \sigma_x^2\right) \\
E[p(t)] &= \exp\left(F_y(t) + \frac{1-e^{-2\lambda_y t}}{4\lambda_y} \sigma_y^2\right)
\end{align*}
\] (10)

while the standard deviations are equal to

\[
\begin{align*}
\sigma[c(t)] &= \exp\left(F_x(t) + \frac{1-e^{-2\lambda_x t}}{4\lambda_x} \sigma_x^2\right) \sqrt{\exp(\sigma_x^2) - 1} = E[c(t)] \sqrt{\exp(\sigma_x^2) - 1} \\
\sigma[p(t)] &= \exp\left(F_y(t) + \frac{1-e^{-2\lambda_y t}}{4\lambda_y} \sigma_y^2\right) \sqrt{\exp(\sigma_y^2) - 1} = E[p(t)] \sqrt{\exp(\sigma_y^2) - 1}
\end{align*}
\] (11)

If the means are time-independent, with \(f_x(t) = f_x\) and \(f_y(t) = f_y\), the following identities hold:

\[
\begin{align*}
F_x(t) &= f_x + (x(0) - f_x) \cdot e^{-\lambda_x t} \\
F_y(t) &= f_y + (y(0) - f_y) \cdot e^{-\lambda_y t}
\end{align*}
\]

2.2. Parameter estimation

Taking a realization of the following process (with constant long run means)
\[
\begin{aligned}
\begin{cases}
    c = e^x \\
p = e^y \\
dx = -\lambda_x (x - f_x) \, dt + \sigma_x \, dW_1 \\
dy = -\lambda_y (y - f_y) \, dt + \sigma_y \left( \rho \, dW_1 + \sqrt{1 - \rho^2} \, dW_2 \right)
\end{cases}
\end{aligned}
\]  

(12)

at discrete time with constant interval of time \( \Delta t \), defined

\[
\begin{aligned}
\begin{cases}
    x_n = x \left( n \Delta t \right) \\
y_n = y \left( n \Delta t \right)
\end{cases}
\end{aligned}
\]

a bivariate autoregressive process with correlated random components is obtained:

\[
\begin{aligned}
\begin{cases}
    x_n = a_x x_{n-1} + b_x + \tilde{\sigma}_x \, \varepsilon_{n,1} \\
y_n = a_y y_{n-1} + b_y + \tilde{\sigma}_y \left( \rho \, \varepsilon_{n,1} + \sqrt{1 - \rho^2} \, \varepsilon_{n,2} \right)
\end{cases}
\end{aligned}
\]  

(13)

with \( \varepsilon_{n,i} \) \( \varepsilon_{m,j} \) independent, normally distributed with mean null and unitary variance. The parameters in equations (13) are related to the parameters in equations (12) by the following identities

\[
\begin{aligned}
a_x &= e^{-\lambda_x \Delta t} \\
b_x &= \left( 1 - e^{-\lambda_x \Delta t} \right) f_x \\
\tilde{\sigma}_x &= \sigma_x \sqrt{\frac{1 - e^{-2\lambda_x \Delta t}}{2\lambda_x}} \\
a_y &= e^{-\lambda_y \Delta t} \\
b_y &= \left( 1 - e^{-\lambda_y \Delta t} \right) f_y \\
\tilde{\sigma}_y &= \sigma_y \sqrt{\frac{1 - e^{-2\lambda_y \Delta t}}{2\lambda_y}}
\end{aligned}
\]

Assuming that \( d\varepsilon_{n,i} \), the interval of uncertainty on \( \varepsilon_{n,i} \), is "small enough", the probability of the sequence of random jumps \( \varepsilon_{1,1}, \varepsilon_{1,2}, \varepsilon_{2,1}, \varepsilon_{2,2}, \varepsilon_{3,1}, \varepsilon_{3,2}, \ldots, \varepsilon_{N,1}, \varepsilon_{N,2} \) is
\[ P(\varepsilon) \; d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n, d\varepsilon_n = \]
\[ = \frac{1}{(2\pi)^N} \exp \left( -\frac{1}{2} \sum_{j=1}^{N} (\varepsilon_{j,1}^2 + \varepsilon_{j,2}^2) \right) d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_n, d\varepsilon_n \]

Given the time series \( \{(p_j, c_j)\}_{j=0,\ldots,N} \) the jumps must be equal to

\[ \left\{ \begin{array}{l}
\varepsilon_{j,1} = \frac{1}{\sigma_a} (\ln c_j - a_x \ln c_{j-1} - b_x) \\
\varepsilon_{j,2} = \frac{1}{\sqrt{1-\rho^2}} \left( \frac{1}{\sigma_y} (\ln p_j - a_y \ln p_{j-1} - b_y) - \frac{\rho}{\sigma_x} (\ln c_j - a_x \ln c_{j-1} - b_x) \right) 
\end{array} \right. \]

then (if the interval of uncertainty \( dc, dp \) are "small enough"), it follows:

\[ P(p, c) \; dc_1 dp_1 dc_2 dp_2 \ldots dc_n, dp_n = \]
\[ = \frac{1}{(2\pi)^N} \exp \left( -\frac{1}{2} \sum_{j=1}^{N} \left( \frac{(\ln c_j - a_x \ln c_{j-1} - b_x)^2}{\sigma_x^2} + \frac{(\ln p_j - a_y \ln p_{j-1} - b_y)^2}{\sigma_y^2} + \right. \right. \\
\left. \left. - 2\rho \frac{(\ln c_j - a_x \ln c_{j-1} - b_x) (\ln p_j - a_y \ln p_{j-1} - b_y)}{\sigma_x \sigma_y} \right) \right) \frac{dc_1 dp_1 dc_2 dp_2 \ldots dc_n dp_n}{\tilde{\sigma}_x N \tilde{\sigma}_y N (1-\rho^2)^{\frac{N}{2}} \prod_{i=1}^{N} c_i p_i} \]

Adopting the criterion of the maximum likelihood, we want to find the parameters \( a_x, b_x, a_y, b_y, \tilde{\sigma}_x, \tilde{\sigma}_y, \rho \) which maximize \( P \).

Taking the logarithm of \( P \) and neglecting constant terms, maximizing \( P \) is the same as minimizing \( L \) with
The problem can be solved numerically solving a fifth order polynomial equation. Mathematical details are in the Appendix.

3. Optimal strategy of an electricity producer

We consider the risk-return trade-off of an electricity producer who can sell part of his capacity at the forward price $p_F$ and the remaining capacity in the spot market at the price $p$. The decision process takes place at time $t = 0$ (now); the forward can have any maturity $T$ from now, but as long as the maturity is set, i.e. the forward has been signed, the producer will sell the whole capacity at time $T$: a portion of it will be devoted to honor the forward, the remaining to the spot market. We will see that the optimal strategy depends, quite reasonably, also on the maturity $T$.

If the producer can neither buy and store the fuel nor buy it forward, his profit function is:

$$ G = \alpha (p_F - c) + \beta \max(p - c, 0) - C_f $$

since he will sell on the spot market only if profitable (i.e. $p > c$).

- $p$ is the spot price of one unit of electricity power at time $t = T$,
- $c$ the (variable) production cost to produce the unit (i.e. a multiple of the cost of fuel) at time $t = T$,
- $p_F$ is the forward price at time $t = 0$ of one unit of electricity power with maturity $T$,
- $C_f$ are fixed costs,
• \( \alpha, \beta \) are the production decision variables: \( \alpha \) is the quantity of energy committed at the fixed price \( p_F \) and \( \beta \) the quantity to be sold in the spot market.

The expected profit and the profit standard deviation are

\[
\left\{ \begin{array}{l}
E[G] = \alpha (p_F - E[c]) + \beta E[\max(p - c, 0)] \\
\sigma[G] = \sqrt{Var[c] \alpha^2 - 2 Cov[\max(p - c), c] \alpha \beta + Var[\max(p - c, 0)] \beta^2}
\end{array} \right.
\]

Since the decision variables are bounded by the capacity constraint

\[ 0 \leq \alpha + \beta \leq Q \]

(where \( Q \) is the maximum production capacity), the producer can usually reduce the risk by changing the allocation of his sales, i.e. by finding the optimal allocation such that the risk is minimized for a given expected profit. For simplicity we assume here that the producer sells the whole production. The more general case, in which the producer can decide to sell a reduced capacity, is investigated in [7]. For more theoretical details in this Section refer again to [7].

In any case the sale is risky, even in case of a total sale on forward \((\alpha = Q, \beta = 0)\), since the uncertainty is given by the cost of input:

\[ \sigma[G]_{\alpha=Q, \beta=0} = Q \sigma[c] \]

In case of a total sale in the spot market \((\alpha = 0, \beta = Q)\)

\[ \sigma[G]_{\alpha=0, \beta=Q} = Q \sigma[\max(p - c, 0)] \]

It has to be noted that such risks are time dependent and depend not only on \( \sigma_x, \sigma_y \) but also on \( \lambda_x, \lambda_y \). So, for example, spot sale may be more risky when a short maturity forward is considered, but less risky when the forward has a longer maturity. From the simulations it will be apparent how the optimal strategy changes with the forward maturity. It has to be noted that the reversion keeps bounded the standard deviations of both \( p \) and \( c \) in the long run.

It is easy to show that forward sale is more profitable than spot sale if

\[ p_F > E[\max(p, c)] \]

In particular the value \( E[\max(p, c)] = E[\max(p - c, 0)] + E(c) \) can be interpreted as a benchmark forward price for a risk neutral producer: under risk
neutrality the value of selling through a forward is equal to the expected production marginal cost plus the expected value of the spark-spread option (opportunity to produce energy when it is economically profitable) embedded in spot selling.

Assuming that the producer sells the whole production\(^2\), the optimal strategies are functions of the following five parameters: \(E[\max(p - c, 0)]\), \(E[\max(p, c)]\), \(Var[\max(p - c, 0)]\), \(Var[\max(p, c)] = Var[\max((p - c, 0) + c)]\), \(Cov[\max(p - c, 0), \max(p, c)]\). The analytical expressions are in Appendix.

The efficient frontier is an arc of hyperbola in the \((\sigma(G), E(G))\) plane. Defining \(v = \alpha/Q\) the fraction of production devoted to the forward, the extremes of the hyperbola correspond to \(v = 1\) and \(v = 0\). As a further useful choice criterion to select the optimal spot/forward combination, for each strategy we must consider the Sharpe Ratio \(SR = \frac{E(G)}{\sigma(G)}\), i.e. the return per unit of risk (usually adopted in the financial literature to grossly order alternative investment opportunities, [16]) and show (see [7]) that an optimal combination at full capacity can be found, called \(v_{ott} \in [0, 1]\) which maximizes \(SR\). All those combinations maximizing \(SR\) (i.e. with \(v = v_{ott}\)) are efficient. At full capacity, only the combinations with \(E[G] > E[G]_{\alpha=\beta=v_{ott}Q}\) are efficient. The optimal choice varies according to the price of the forward, its maturity and the various parameters involved in the two stochastic processes governing spot price and fuel. In particular, a higher correlation usually allows a greater risk reduction opportunities by diversification of the sales.

### 4. Optimal timing in plant investment

We consider the case of a producer seeking to maximize the expected value of a producing plant depending on the time of investment. Following [13] we assume that at time \(t = 0\) a project is postponed up to a time \(t^*\) if \(t^* > 0\) is such that the discounted expected value of the cash flow generated by the project reaches its maximum in \([0, +\infty]\). A profit maximizer decision maker is supposed to adopt the more profitable choice. Consider the following maximization problem:

\[
\max_{t_0 \in [0, +\infty]} V(t_0) = -C(Q) e^{-r t_0} + \int_{t_0}^{+\infty} E[Q \max(p_t - c_t, 0) | \mathcal{F}_0] e^{-r t} dt
\]

\(^2\)In [7] the case of the producer willing to sell less than the total production is fully studied.
where

- $Q$ is the capacity of the plant
- $C(Q)$ is the investment cost (which is supposed to be constant and known)
- $p_t$ is the electricity spot price per unit of time at time $t$
- $c_t$ is the input spot price per unit of time at time $t$
- $F_0$ is the information available at time $t = 0$
- $r$ is the discount rate
- $t_0$ is the investment time.

Let $t^*$ be the value of $t_0$ maximizing $V(t_0)$: the project will be postponed if $t^* > 0$.

The derivative of $V$ is

\[
\frac{dV}{dt_0} = r C(Q) e^{-rt_0} - E[Q \max(p_{t_0} - c_{t_0}, 0) | F_0] e^{-rt_0} = (r C(Q) - Q E[\max(p_{t_0} - c_{t_0}, 0) | F_0]) e^{-rt_0}
\]

and implies that

\[
\frac{dV}{dt_0} < 0 \iff E[\max(p_{t_0} - c_{t_0}, 0) | F_0] > r \frac{C(Q)}{Q}
\]

From this condition, if the project is postponed, the capital $C(Q)$ can be invested riskless at a rate $r$ and it generates (in an infinitesimal interval of time $\delta t$) a profit equal to $r C(Q) \delta t$, otherwise the project is expected to generate a profit equal to $Q E[\max(p_{t_0}^\tau - c_{t_0}^\tau, 0) | F_0] \delta t$. Rearranging equation (15) we obtain

\[
E[\max(p_{t_0}, c_{t_0}) | F_0] > E[c_{t_0} | F_0] + r \frac{C(Q)}{Q}
\]

i.e. it is not convenient to postpone after $t_0$ if the revenue from the plant investment in $t_0$ is higher than the cost of input plus the revenue from investing the investment cost at the risk free rate.
A necessary condition for optimality is

\[ V(t^*) > 0 \iff \int_{t^*}^{+\infty} E[\max(p_t - c_t, 0) | \mathcal{F}_0] e^{-rt} dt > \frac{C(Q)}{Q} e^{-r t^*} \]

In particular, if the condition

\[ \int_{0}^{+\infty} E[\max(p_t - c_t, 0) | \mathcal{F}_0] e^{-rt} dt > \frac{C(Q)}{Q} \] (16)

is not satisfied the project will surely be rejected. Inequality 16 states that unit (per megawatt) discounted expected profit must be higher than the unit investment cost.

When \( r = 0 \), condition 15 implies that \( \frac{dV}{dt_0} < 0 \) for every \( t_0 > 0 \), then the investment project is immediately activated if

\[ V(0) > 0 \iff \int_{0}^{+\infty} E[\max(p_t - c_t, 0) | \mathcal{F}_0] dt > \frac{C(Q)}{Q} \]

Conversely, unless there is a positive trend in prices (i.e. \( p(t) \) is supposed to increase indefinitely), the function \( E[\max(p_t - c_t, 0) | \mathcal{F}_0] \) (as a function of \( t \)) is upper bounded, so there is an \( r \) such that \( \frac{dV}{dt_0} > 0 \) for every \( t \), that is: there is surely a minimal interest rate \( r \) such that the project is always postponed.

For continuous processes \( p_t \) and \( c_t \) (as we assume here) the maximum of \( V \) will be either in \( t = 0 \) or when its derivative is zero. To obtain an immediate start of the project a necessary condition is that at time \( t = 0 \)

\[ \frac{dV}{dt_0} \bigg|_{t_0=0} = r C(Q) - Q \max(p_0 - c_0, 0) \leq 0 \]

which solves in \( p_0 \geq c_0 + r \frac{C(Q)}{Q} \), making use of the obvious additional condition that \( p_0 > c_0 \). To this purpose it is additionally required that at possible future times \( \tilde{t} \) of local maxima, that is

\[ \frac{dV}{dt} \bigg|_{t=\tilde{t}} = 0 \iff E[\max(p_{\tilde{t}} - c_{\tilde{t}}, 0) | \mathcal{F}_0] = r \frac{C(Q)}{Q} \]
\( V(t) \leq V(0) \) holds. If any of these two conditions do not hold, we will have \( t^* = \tilde{t} > 0 \).

We have obtained an analytical form for \( E[\max (p_t - c_t, 0) | F_0] \):

\[
E[\max (p_t - c_t, 0) | F_0] = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{A(t)}{\sqrt{B(t)}} + \frac{\sqrt{B(t)}}{2} \right) \right) e^{A(t)} +
\left. \left( 1 + \text{erf} \left( \frac{A(t)}{\sqrt{B(t)}} - \frac{\sqrt{B(t)}}{2} \right) \right) \right) E[c_t|F_0]
\]

where

\[
\begin{align*}
A(t) &= \ln \left( \frac{E[p_t|F_0]}{E[c_t|F_0]} \right) \\
B(t) &= \frac{1}{2} \left( \sigma^2_\xi(t) + \sigma^2_\varphi(t) - 2 \rho \sigma_\xi(t) \sigma_\varphi(t) \right)
\end{align*}
\]

and the problem can be solved numerically.

5. Empirical application

5.1. German market in year 2005

We consider daily prices (in particular the timezone 9.00-10.00) of power in German market (EEX) and one-day forward gas prices for the year 2005. Since both series show no significant trends (see figures 5 and 4) the long run mean can be taken as a constant for both processes.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \lambda_x )</td>
<td>0.035 day(^{-1} )</td>
</tr>
<tr>
<td>( \lambda_y )</td>
<td>0.27 day(^{-1} )</td>
</tr>
<tr>
<td>( f_x )</td>
<td>3.94</td>
</tr>
<tr>
<td>( f_y )</td>
<td>4.09</td>
</tr>
<tr>
<td>( \sigma_x )</td>
<td>0.1</td>
</tr>
<tr>
<td>( \sigma_y )</td>
<td>0.26</td>
</tr>
<tr>
<td>( \rho )</td>
<td>9.9%</td>
</tr>
</tbody>
</table>

Table 1: parameter estimates for the processes of electricity and gas daily rates of returns

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} \, dt = 2 \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\sqrt{2}x} e^{-t^2} \, dt - 1 = 2 \text{csn}(\sqrt{2}x) - 1
\]

where \( \text{csn} \) is cumulative standard normal function.
Figure 4: EEX power price (in €/MWh) relatively to the hour 9.00-10.00 for the year 2005.

Figure 5: One day forward gas price (in €/MWh equiv, assuming the standard efficiency of 39%) for the year 2005.
Figure 6: Log electricity prices (Jan-Nov 2006) and expectations.

Figure 7: Log gas prices (Jan-Nov 2006) and expectations.
The parameters maximizing the likelihood are reported in Table (1).

In Figure (6) and Figure (7) we plot the log price of EEX electricity and the 1-day forward gas log price from January to November 2006 against the $\pm \sigma$, $\pm 2\sigma$ and $\pm 3\sigma$ boundaries around the respective expectations evaluated on December 30, 2005.

5.2. Efficient risk management for a german electricity producer

An empirical application follows in which the optimal decision problem for a gas plant is simulated. The parameters obtained by the estimation on German EEX electricity spot prices for year 2005 are taken as a reference for the historical simulation. We build the efficient pairs $(\alpha, \beta)$ in the mean/variance plane of the profit function and identify an optimal solution adopting the Sharpe Ratio as an optimization criterion. A sensitivity analysis is performed on the effect of the correlation coefficient $\rho$ on the optimal strategy, as observed in different time spans.

Among the various simulations we present some of them in which the effect of the forward maturity is apparent.

Note that, since $\lambda_x = 0.035$ and $\lambda_y = 0.27$, the transitory effects due to $c(0)$ and $p(0)$ are dampened at 37% in about $\frac{1}{\lambda_x} = \frac{1}{0.035} = 28.6$ and $\frac{1}{\lambda_y} = \frac{1}{0.27} = 3.7$ days respectively (i.e. the dependence on $c(0)$ and $p(0)$ vanishes in about 2–3 months and 1–1.5 weeks respectively).

In Figure (8) $v_{opt}$ (i.e. the optimal fraction of production to allocate to a forward with maturity $t$ (days) according to the Sharpe Ratio criterion) as a function of its price $p_F$ and $t$ is plotted. In Table (2) we reported the value of $v_{opt}$ for some values of $p_F$ and $t$. The parameters obtained by the analysis on the german market reported in Table (1) and time 0 prices equal to their long run averages ($c(0) = E[c(\infty)] = 55.23\, \text{€}/\text{MWh}$, $p(0) = E[p(\infty)] = 63.6\, \text{€}/\text{MWh}$) have been used.

<table>
<thead>
<tr>
<th>$t$</th>
<th>$p_F$</th>
<th>60</th>
<th>69</th>
<th>70</th>
<th>71</th>
<th>80</th>
</tr>
</thead>
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<td>1</td>
<td></td>
<td>0.725</td>
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<td>0.958</td>
<td>0.966</td>
<td>1</td>
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<tr>
<td>7</td>
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<td>0.734</td>
<td>0.761</td>
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</tr>
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<td></td>
<td>0</td>
<td>0.400</td>
<td>0.441</td>
<td>0.480</td>
<td>0.746</td>
</tr>
</tbody>
</table>

Table 2: Optimal SR proportion in forward selling
Reading Table (2) by columns, it is apparent that spot selling becomes more interesting as long as the forward maturity increases. As long as the forward price decreases, for each maturity spot selling becomes the optimal choice (not reported in Table). This gives a clear idea that a forward must be signed with care, considering the effect that the maturity can have on the diversification.

As a further analysis in Figure (9) we report a graph showing the efficient \((\sigma(G), E(G))\) frontiers for given maturities \((p_f = 69\text{€}/\text{MWh})\).

For \(T > 90\) there is not a significant difference between the efficient frontiers because of the convergence to their long run averages of all the parameters.

In Figure (10) a correlation of 80% has been used for comparison with \(\rho = 9.9\%\) in Figure(9). It is clear that the opportunity of risk reduction by sale diversification has increased. Comparing Figure (11) with Figure (12) we observe that \(E[\max(p,c)]\) (the benchmark forward price for a risk neutral producer) may be strongly affected by the correlation: a high correlation usually reduces the expected profit of the spot sale, this is the reason why \(E[\max(p,c)]\) is quite lower with \(\rho = 80\%\).
Figure 9: Efficient frontiers for different maturities.

Figure 10: Efficient frontiers if $\rho = 0.8$. 
Figure 11: The benchmark price $E[\max(p, c)]$ as a function of the maturity $T$.

Figure 12: The benchmark price $E[\max(p, c)]$ as a function of the maturity $T$ if the correlation were $\rho = 80\%$. 
5.3. Optimal timing in plant investment for a german electricity producer

Starting from the expression of the objective function (the time 0 plant value) in eq. (14), we apply the change of variable \( \tau = e^{-rt} \) and obtain the following definite integral:

\[
V(\tau_0) = \frac{Q}{r} \int_0^{\tau_0} \left[ g(\tau) - \frac{r C(Q)}{Q} \right] d\tau
\]

where

\[
g(\tau) = \frac{1}{2} \tilde{c}(\tau) \left( \left( 1 + \text{erf} \left( \frac{\tilde{A}(\tau)}{\sqrt{\tilde{B}(\tau)}} + \frac{\sqrt{\tilde{B}(\tau)}}{2} \right) \right) e^{\tilde{A}(\tau)} + \right.
\]

\[
- \left( 1 + \text{erf} \left( \frac{\tilde{A}(\tau)}{\sqrt{\tilde{B}(\tau)}} - \frac{\sqrt{\tilde{B}(\tau)}}{2} \right) \right)
\]

\[
\tilde{A}(\tau) = \tilde{y}(\tau) - \tilde{x}(\tau) + \frac{1}{2} \tilde{\sigma}_\phi^2(\tau) - \frac{1}{2} \tilde{\sigma}_\xi^2(\tau)
\]

\[
\tilde{B}(\tau) = \frac{1}{2} \left( \tilde{\sigma}_\xi^2(\tau) + \tilde{\sigma}_\phi^2(\tau) - 2 \rho \tilde{\sigma}_\xi(\tau) \tilde{\sigma}_\phi(\tau) \right)
\]

\[
\tilde{x}(\tau) = f_x + (x_0 - f_x) \tau \frac{\lambda_x}{\lambda_y}
\]

\[
\tilde{y}(\tau) = f_y + (y_0 - f_y) \tau \frac{\lambda_y}{\lambda_y}
\]

\[
\tilde{\sigma}_\xi(\tau) = \frac{\sigma_x}{\sqrt{2\lambda_y}} \sqrt{1 - \tau^\frac{2\lambda_y}{\lambda_x}}
\]

\[
\tilde{\sigma}_\phi(\tau) = \frac{\sigma_y}{\sqrt{2\lambda_x}} \sqrt{1 - \tau^\frac{2\lambda_x}{\lambda_y}}
\]

\[
\tilde{c}(\tau) = \exp \left( \tilde{x}(\tau) + \frac{1}{2} \tilde{\sigma}_\xi^2(\tau) \right)
\]

and \( \tau_0 = \exp(-rt_0) \in (0, 1) \).

\( g(\tau) \) in Eqn. (17) is a transformation of the expected unit profit generated by the plant for \( \tau \) ranging from 0 to 1. In particular observe that

\[
g(0) = \lim_{t \to \infty} E \left[ \max(p_t - c_t, 0) \right]
\]

(i.e. the expected unit profit in the long run) and \( g(1) = \max(p_0 - c_0, 0) \), so as a consequence of the time transformation \( g \) actually ranges from an infinite...
future time \((\tau = 0)\) to time 0 \((\tau = 1)\). For the german market data we obtain \(g(0) = 15.77\,\text{€}/\text{MW}\). In the following analysis we will assume \(p_0 = 65\,\text{€}/\text{MW}\) and \(c_0 = 55\,\text{€}/\text{MW}\). Since \(g\) depends on many parameters it is difficult to comment its general behaviour. However, based on the process estimates in table (1) for the german market, we observe that the function \(g(\tau)\) remains constant for the majority of its domain (see for example Figure (13) where \(g(\tau)\) is plotted for parameter values as in table (1) and adopting interest rate \(r\) as large as \(5 \cdot 10^{-4}\,\text{day}^{-1}\) \((\approx 0.2\,\text{year}^{-1})\)). This feature depends mainly on the assumption that the long run averages of gas and electricity prices are constant and that the speeds of reversion for two processes \((\lambda_x \text{ and } \lambda_y)\) are high with respect to \(r\).

The integral in (17) has in general an almost definitive sign depending on the relative values of \(g(0)\) and \(\frac{rC(Q)}{Q}\). Under these conditions decision to start the project will in general be of the type "do it within some weeks or never". If the opportunity cost of investing is always lower than the expected profit, that is

\[
g(\tau) > \frac{rC(Q)}{Q}, \quad \text{for all } \tau \in [0, 1]
\]

then \(V(\tau_0)\) will obviously be maximum for \(\tau_0 = 1\), that is \(t^* = 0\). If (due to time 0 level of prices) this inequality is reversed in some neighbor \([\tau^*, 1]\) as is the case in Figure (13) then the investing decision is delayed until (in \(\tau^*\)) it is restored in the original versus. In the same figure we see that for initial prices \(p_0\) and \(c_0\)
and \( \frac{rC(Q)}{Q} = 0.8 g(0) = 12.77 \), we have \( \tau^* \simeq 0.99 \) (corresponding to an optimal expected delay \( t^* \simeq 2.5 \) months). If inequality (19) is reversed for all \( \tau \in [0, 1] \), the project will never start. It is interesting to notice the relevant impact of the real option value embedded in the plant unit payoff, that is \( \max (p_t - c_t, 0) \). Observe to this purpose that in general \( E [\max (p_t - c_t, 0)] > E [p_t] - E [c_t] \). If a decision maker gives up a project after observing that \( E [p_t] - E [c_t] \leq \frac{rC(Q)}{Q} \) (expected profit is less or equal the opportunity cost of the investment), he would not adopt an optimal decision as he overlooks the option to stop the plant at any time \( t \) if economic conditions are not satisfactory. It therefore can happen that for a given plant project we have \( E [\max (p_t - c_t, 0)] > \frac{rC(Q)}{Q} \geq E [p_t] - E [c_t] \), in which case plant adoption is the right decision.

The difference \( E [\max (p_t - c_t, 0)] - \frac{rC(Q)}{Q} \) depends in a large portion on the diffusion and the correlation parameters of \( x \) and \( y \). In particular it increases the greater is the variance of the two processes and the lower is the correlation between them (a result which is coherent with the literature on the theory of options). The figure (14) shows the graph of \( g(\rho, \tau) \).

The correlation between \( x \) and \( y \) has a relevant impact on the value of plant project, and consequently on the adoption/refusal decision. The impact of this parameter is expected as it appears in the function \( \tilde{B} (\tau) \) in (18). However its relevance may change from case to case depending on several circumstances (e.g. if functions \( \tilde{\sigma}_x (\tau) \) and \( \tilde{\sigma}_y (\tau) \) have large different values the impact of \( \rho \) will tend to vanish). Setting, for example, the level \( \frac{rC(Q)}{Q} = 9 \) (corresponding to the lower bold line in Figure (14)) helps to distinguish the refusal decision area from the acceptance one. The upper bold line corresponds to the graph of \( g(\tau) \) for a correlation parameter \( \rho = 0.099 \), that is the case of german market previously cosidered in Figure (13).

Going into major details of the german market data we calculate several important decision variables. In this discussion we assume an annual interest rate \( r = 0.05 \). So \( \lambda_y \) and \( \lambda_x \) are "large" with respect to \( r \). The asymptotic value of \( g \) can be calculated through the following expression:
Figure 14: Graph of $g(\rho, \tau)$ for $p_0 = 65$ and $c_0 = 55$. Other parameters are set equal to estimated parameters for German market.

$$g(0) = \frac{1}{2} \tilde{c}(0) \left( \left( 1 + \text{erf} \left( \frac{\tilde{A}(0)}{\sqrt{\tilde{B}(0)}} + \frac{\sqrt{\tilde{B}(0)}}{2} \right) \right) e^{\tilde{A}(0)} + \\ - \left( 1 + \text{erf} \left( \frac{\tilde{A}(0)}{\sqrt{\tilde{B}(0)}} - \frac{\sqrt{\tilde{B}(0)}}{2} \right) \right) \right)$$

where

$$\begin{align*}
\tilde{A}(0) &= f_y - f_x + \frac{\sigma_y^2}{4\lambda_y} - \frac{\sigma_x^2}{4\lambda_x} \\
\tilde{B}(0) &= \frac{1}{4} \left( \frac{\sigma_x^2}{\lambda_x} + \frac{\sigma_y^2}{\lambda_y} - 2\rho \frac{\sigma_x \sigma_y}{\sqrt{\lambda_x \lambda_y}} \right) \\
\tilde{c}(0) &= \exp \left( f_x + \frac{\sigma_x^2}{4\lambda_x} \right)
\end{align*}$$
Then (neglecting the "short lived" deviations near \( \tau = 1 \) due to time 0 prices and the quick increase of the volatilities)

\[
V(t_0) \simeq \frac{Q}{r} \left( g(0) - \frac{r}{Q} C(Q) \right) e^{-r.t_0}
\]

For the case of german market data, we can calculate a decision value to accept/refuse a project of a gas fueled plant: such a plant should be rejected if the unit (per megawatt of production capacity) cost of the investment \( \left( \frac{C(Q)}{Q} \right) \) is\(^4\):

\[
\frac{C(Q)}{Q} > \frac{g(0)}{r} = \frac{15.77}{5.6 \cdot 10^{-6}} = 2.8161 \cdot 10^6 \text{ €/MW}
\]

We have already observed in fig. (13) that this project should be delayed, if time 0 prices of gas and electricity are closed to their long run average (i.e. \( p_0 = 65 \), \( c_0 = 55 \)). Under these conditions and fixing \( \frac{rC(Q)}{Q} = 12.77 \), the percentage difference \( \frac{V_r - V_0}{V_r} \) resulting from delaying the project for 75 days is equal to 0.4%, which is not a relevant value.

6. Conclusions

Two problems of considerable importance in corporate risk management in the energetic sector have been investigated: risk hedging through forward selling and optimal timing in plant investment. In both cases, assuming that price and costs are modelled by two related stochastic processes and through proper simulations we find reasonable results, very helpful, in our opinion, to help the decision maker to optimize his choices. Both models can be further extended: for instance in risk hedging one can introduce forward on the input as well.

Correlation in the stochastic component of the gas and electricity price processes can play a relevant role in the two decision problem analysed here, even though its impact changes in the two cases. A positive correlation helps reducing risk in production decisions, since spot selling of electricity becomes less risky. On the other hand a negative correlation increases the value of investing in new production plants, as it increases the real option value embedded in electricity production.

\(^4\)Such a value is based on the estimation of a peak hour of german electricity price. Since only about 6-8 hours are peak hours, the limit cost of investment is actually about \( \frac{1}{4} \) of the value calculated, i.e.: \(~700000e/MW\).
The effect of correlation is worth a deeper analysis. It is common knowledge that economic links between different sectors are the cause of the correlation between prices of different commodities. So in the model a functional relationship between the deterministic components of the processes could be introduced next to the correlation between the stochastic disturbances.

7. References

References


8. Appendix

8.1. Likelihood maximization

Given the objective function

\[ L = \frac{1}{2} \sum_{j=1}^{N} \left( \frac{(\ln c_j - a_x \ln c_{j-1} - b_x)^2}{\tilde{\sigma}_x^2} + \frac{(\ln p_j - a_y \ln p_{j-1} - b_y)^2}{\tilde{\sigma}_y^2} \right) + 2\rho \frac{(\ln c_j - a_x \ln c_{j-1} - b_x)(\ln p_j - a_y \ln p_{j-1} - b_y)}{\tilde{\sigma}_x \tilde{\sigma}_y} \]

\[ + N \ln (\tilde{\sigma}_x) + N \ln (\tilde{\sigma}_y) + \frac{N}{2} \ln (1 - \rho^2) \]

Let

\[
\begin{cases}
  s_x = \frac{1}{\tilde{\sigma}_x \sqrt{1-\rho^2}} \\
  s_y = \frac{1}{\tilde{\sigma}_y \sqrt{1-\rho^2}}
\end{cases}
\]

\( L \) becomes

\[ L = \frac{1}{2} \sum_{j=1}^{N} \left( s_x^2 (\ln c_j - a_x \ln c_{j-1} - b_x)^2 + s_y^2 (\ln p_j - a_y \ln p_{j-1} - b_y)^2 + \right. \]

\[ -2\rho s_x s_y (\ln c_j - a_x \ln c_{j-1} - b_x)(\ln p_j - a_y \ln p_{j-1} - b_y) \left. \right) \]

\[ - N \ln (s_x) - N \ln (s_y) - \frac{N}{2} \ln (1 - \rho^2) \]

As a function of \( a_x, b_x, a_y, b_y \) the optimal \( s_x, s_y, \rho \) must solve the equations

\[ \frac{\partial L}{\partial s_x} = \frac{\partial L}{\partial s_y} = \frac{\partial L}{\partial \rho} = 0 \]

i.e.
\[
\begin{align*}
\frac{\partial L}{\partial s_x} &= s_x \sum_{j=1}^{N} (\ln c_j - a_x \ln c_{j-1} - b_x)^2 + \\
- \rho s_y \sum_{j=1}^{N} (\ln c_j - a_x \ln c_{j-1} - b_x) (\ln p_j - a_y \ln p_{j-1} - b_y) - \frac{N}{s_y} = 0 \\
\frac{\partial L}{\partial s_y} &= s_y \sum_{j=1}^{N} (\ln p_j - a_y \ln p_{j-1} - b_y)^2 \\
- \rho s_x \sum_{j=1}^{N} (\ln c_j - a_x \ln c_{j-1} - b_x) (\ln p_j - a_y \ln p_{j-1} - b_y) - \frac{N}{s_y} = 0 \\
\frac{\partial L}{\partial \rho} &= -s_x s_y \sum_{j=1}^{N} (\ln c_j - a_x \ln c_{j-1} - b_x) (\ln p_j - a_y \ln p_{j-1} - b_y) + N \frac{\rho}{1-\rho^2} = 0
\end{align*}
\]

(20)

\[
\begin{align*}
\frac{\partial L}{\partial s_x} &= s_x \sum_{j=1}^{N} (\ln c_j - a_x \ln c_{j-1} - b_x)^2 \\
- \rho s_y \sum_{j=1}^{N} (\ln c_j - a_x \ln c_{j-1} - b_x) (\ln p_j - a_y \ln p_{j-1} - b_y) - \frac{N}{s_y} = 0 \\
\frac{\partial L}{\partial s_y} &= s_y \sum_{j=1}^{N} (\ln p_j - a_y \ln p_{j-1} - b_y)^2 \\
- \rho s_x \sum_{j=1}^{N} (\ln c_j - a_x \ln c_{j-1} - b_x) (\ln p_j - a_y \ln p_{j-1} - b_y) - \frac{N}{s_y} = 0 \\
\frac{\partial L}{\partial \rho} &= -s_x s_y \sum_{j=1}^{N} (\ln c_j - a_x \ln c_{j-1} - b_x) (\ln p_j - a_y \ln p_{j-1} - b_y) + N \frac{\rho}{1-\rho^2} = 0
\end{align*}
\]

(21)

As a function of the remaining parameters \( L \) becomes

\[
L = N - N \ln (N) + \frac{N}{2} \ln \left( \sum_{i=1}^{N} (\ln c_i - a_x \ln c_{i-1} - b_x)^2 \sum_{j=1}^{N} (\ln p_j - a_y \ln p_{j-1} - b_y)^2 + \\
- \left[ \sum_{k=1}^{N} (\ln c_k - a_x \ln c_{k-1} - b_x) (\ln p_k - a_y \ln p_{k-1} - b_y) \right]^2 \right)
\]
whose minimization is equivalent to minimizing

\[
\tilde{L} = \sum_{i=1}^{N} (\ln c_i - a_x \ln c_{i-1} - b_x)^2 \sum_{j=1}^{N} (\ln p_j - a_y \ln p_{j-1} - b_y)^2 + \\
- \left[ \sum_{k=1}^{N} (\ln c_k - a_x \ln c_{k-1} - b_x) (\ln p_k - a_y \ln p_{k-1} - b_y) \right]^2
\]

Equating to zero the derivatives of \( \tilde{L} \) with respect to \( a_x, b_x, a_y, b_y \) we obtain
\[
\frac{\partial \tilde{L}}{\partial a_x} = -2 \sum_{i=1}^{N} \left[ (\ln c_i - a_x \ln c_{i-1} - b_x) \ln c_{i-1} \right] \sum_{j=1}^{N} \left( \ln p_j - a_y \ln p_{j-1} - b_y \right)^2 + \\
+2 \sum_{i=1}^{N} \left[ (\ln c_i - a_x \ln c_{i-1} - b_x) \ln p_{i-1} \ln p_{j-1} \right] \cdot \\
\sum_{j=1}^{N} \left[ (\ln p_j - a_y \ln p_{j-1} - b_y) \ln c_{j-1} \right] = 0
\]

\[
\frac{\partial \tilde{L}}{\partial a_y} = -2 \sum_{i=1}^{N} \left[ (\ln c_i - a_x \ln c_{i-1} - b_x) \ln c_{i-1} \right] \sum_{j=1}^{N} \left( \ln p_j - a_y \ln p_{j-1} - b_y \right) + \\
+2 \sum_{i=1}^{N} \left[ (\ln c_i - a_x \ln c_{i-1} - b_x) \ln p_{i-1} \ln p_{j-1} \right] \cdot \\
\sum_{j=1}^{N} \left[ (\ln p_j - a_y \ln p_{j-1} - b_y) \ln p_{j-1} \right] = 0
\]

\[
\frac{\partial \tilde{L}}{\partial b_x} = -2 \sum_{i=1}^{N} \left[ (\ln c_i - a_x \ln c_{i-1} - b_x) \ln c_{i-1} \right] \sum_{j=1}^{N} \left( \ln p_j - a_y \ln p_{j-1} - b_y \right)^2 + \\
+2 \sum_{i=1}^{N} \left[ (\ln c_i - a_x \ln c_{i-1} - b_x) \ln p_{i-1} \ln p_{j-1} \right] \cdot \\
\sum_{j=1}^{N} \left[ (\ln p_j - a_y \ln p_{j-1} - b_y) \ln p_{j-1} \right] = 0
\]

\[
\frac{\partial \tilde{L}}{\partial b_y} = -2 \sum_{i=1}^{N} \left[ (\ln c_i - a_x \ln c_{i-1} - b_x) \ln c_{i-1} \right] \sum_{j=1}^{N} \left( \ln p_j - a_y \ln p_{j-1} - b_y \right) + \\
+2 \sum_{i=1}^{N} \left[ (\ln c_i - a_x \ln c_{i-1} - b_x) \ln p_{i-1} \ln p_{j-1} \right] \cdot \\
\sum_{j=1}^{N} \left[ (\ln c_j - a_x \ln c_{j-1} - b_x) \ln p_{j-1} \right] = 0
\]

Last two equations are solved by
\[
\begin{align*}
\sum_{i=1}^{N} (\ln c_i - a_x \ln c_{i-1} - b_x) &= 0 \\
\sum_{j=1}^{N} (\ln p_j - a_y \ln p_{j-1} - b_y) &= 0
\end{align*}
\]

\[
\Leftrightarrow \quad \begin{align*}
b_x &= \frac{1}{N} \sum_{j=1}^{N} \ln c_j - a_x \frac{1}{N} \sum_{j=1}^{N} \ln c_{j-1} \\
b_y &= \frac{1}{N} \sum_{j=1}^{N} \ln p_j - a_y \frac{1}{N} \sum_{j=1}^{N} \ln p_{j-1}
\end{align*}
\]

Defining

\[
\begin{align*}
\Delta_{(1)i} &= \ln c_{i-1} - \frac{1}{N} \sum_{j=1}^{N} \ln c_{j-1} \\
\Delta_{(2)i} &= \ln c_i - \frac{1}{N} \sum_{j=1}^{N} \ln c_j \\
\Delta_{(3)i} &= \ln p_{i-1} - \frac{1}{N} \sum_{j=1}^{N} \ln p_{j-1} \\
\Delta_{(4)i} &= \ln p_i - \frac{1}{N} \sum_{j=1}^{N} \ln p_j
\end{align*}
\]

\[
\begin{align*}
\gamma_1 &= \sum_{i=1}^{N} \Delta_{(1)i}^2 \\
\gamma_2 &= \sum_{i=1}^{N} \Delta_{(1)i} \Delta_{(2)i} \\
\gamma_3 &= \sum_{i=1}^{N} \Delta_{(2)i}^2 \\
\gamma_4 &= \sum_{i=1}^{N} \Delta_{(1)i} \Delta_{(3)i} \\
\gamma_5 &= \sum_{i=1}^{N} \Delta_{(1)i} \Delta_{(4)i} \\
\gamma_6 &= \sum_{i=1}^{N} \Delta_{(2)i} \Delta_{(3)i} \\
\gamma_7 &= \sum_{i=1}^{N} \Delta_{(2)i} \Delta_{(4)i} \\
\gamma_8 &= \sum_{i=1}^{N} \Delta_{(3)i}^2 \\
\gamma_9 &= \sum_{i=1}^{N} \Delta_{(3)i} \Delta_{(4)i} \\
\gamma_{10} &= \sum_{i=1}^{N} \Delta_{(4)i}^2
\end{align*}
\]

First two equations become

\[
\begin{align*}
\sum_{i=1}^{N} [ (\Delta \ln c_i - a_x \Delta_{(1)i}) \Delta_{(1)i} ] \sum_{j=1}^{N} (\Delta_{(4)j} - a_y \Delta_{(3)j})^2 &= \\
= \sum_{i=1}^{N} [ (\Delta_{(2)i} - a_x \Delta_{(1)i}) (\Delta_{(4)i} - a_y \Delta_{(3)i}) ] \sum_{j=1}^{N} [ (\Delta_{(4)j} - a_y \Delta_{(3)j}) (\Delta_{(1)j}) ] \\
\sum_{i=1}^{N} (\Delta_{(2)i} - a_x \Delta_{(1)i})^2 \sum_{j=1}^{N} [ (\Delta_{(4)j} - a_y \Delta_{(3)j}) \Delta_{(3)j} ] &= \\
= \sum_{i=1}^{N} [ (\Delta_{(2)i} - a_x \Delta_{(1)i}) (\Delta_{(4)i} - a_y \Delta_{(3)i}) ] \sum_{j=1}^{N} [ (\Delta_{(2)j} - a_x \Delta_{(1)j}) \Delta_{(3)j} ]
\end{align*}
\]
The first equation is

\[
(\gamma_1 \gamma_8 - \gamma_4^2) a_x a_y^2 + 2 (\gamma_4 \gamma_5 - \gamma_1 \gamma_9) a_x a_y + (\gamma_4 \gamma_6 - \gamma_2 \gamma_8) a_y^2 + \\
+ (\gamma_1 \gamma_0 - \gamma_5^2) a_x + (2 \gamma_2 \gamma_9 - \gamma_4 \gamma_7 - \gamma_5 \gamma_6) a_y + (\gamma_5 \gamma_7 - \gamma_2 \gamma_0) = 0
\]

The second equation is

\[
(\gamma_1 \gamma_8 - \gamma_4^2) a_x^2 a_y + (\gamma_4 \gamma_5 - \gamma_1 \gamma_9) a_x^2 + 2 (\gamma_4 \gamma_6 - \gamma_2 \gamma_8) a_x a_y + \\
+ (2 \gamma_2 \gamma_9 - \gamma_4 \gamma_7 - \gamma_5 \gamma_6) a_x + (\gamma_3 \gamma_8 - \gamma_6^2) a_y + (\gamma_6 \gamma_7 - \gamma_3 \gamma_9) = 0
\]

From the first equation we obtain

\[
a_x = \frac{-(\gamma_4 \gamma_6 - \gamma_2 \gamma_8) a_y^2 + (2 \gamma_2 \gamma_9 - \gamma_4 \gamma_7 - \gamma_5 \gamma_6) a_y + (\gamma_5 \gamma_7 - \gamma_2 \gamma_0)}{(\gamma_1 \gamma_8 - \gamma_4^2) a_y^2 + 2 (\gamma_4 \gamma_6 - \gamma_1 \gamma_9) a_y + (\gamma_1 \gamma_0 - \gamma_5^2)}
\]

Replacing \(a_x\) into the second equation a fifth degree polynomial equation for \(a_y\) is obtained. It can be numerically solved (it surely has at least one solution).

### 8.2. Mathematical formulas for the optimal strategy of an electricity producer

Let

\[
\begin{align*}
\bar{x} &= F_x(t) = f_x + (\ln(c_0) - f_x) e^{-\lambda_x t} \\
\bar{y} &= F_y(t) = f_y + (\ln(p_0) - f_y) e^{-\lambda_y t} \\
\sigma_x &= \sqrt{\frac{1-\exp(-2 \lambda_x t)}{2 \lambda_x}} \\
\sigma_y &= \sqrt{\frac{1-\exp(-2 \lambda_y t)}{2 \lambda_y}} \\
\bar{c} &= E[c] = \exp(\bar{x} + \frac{\sigma_x^2}{2}) \\
\bar{p} &= E[p] = \exp(\bar{y} + \frac{\sigma_y^2}{2})
\end{align*}
\]

The parameters determining the efficient frontier are related to the parameters of the process as follows
\[
\begin{align*}
E[c] &= \int_{\mathbb{R}^2} e^x P(x, y) \, dx \, dy = \\
&= \bar{c} = \exp \left( \bar{x} + \frac{\sigma_x^2}{2} \right) \\
E[\max (p - c, 0)] &= \int_{\mathbb{R}^2} \max (e^y - e^x, 0) P(x, y) \, dx \, dy = \\
&= \frac{1}{2} \left( 1 + \text{erf} \left( \frac{\bar{y} - \bar{x} + \sigma_\varphi (\sigma_\varphi - \rho \sigma_\xi)}{\sqrt{2} (\sigma_\xi^2 + \sigma_\varphi^2 - 2\rho \sigma_\xi \sigma_\varphi)} \right) \right) \bar{p} + \\
&- \frac{1}{2} \left( 1 + \text{erf} \left( \frac{\bar{y} - \bar{x} + \sigma_\xi (\rho \sigma_\varphi - \sigma_\xi)}{\sqrt{2} (\sigma_\xi^2 + \sigma_\varphi^2 - 2\rho \sigma_\xi \sigma_\varphi)} \right) \right) \bar{c} \\
\text{Var}[\max (p - c, 0)] &= \int_{\mathbb{R}^2} (\max (e^y - e^x, 0))^2 P(x, y) \, dx \, dy - (E[\max (p - c, 0)])^2 = \\
&= \bar{p}^2 \exp (\sigma_\varphi^2) \left( 1 + \text{erf} \left( \frac{\bar{y} - \bar{x} + 2\sigma_\varphi (\sigma_\varphi - \rho \sigma_\xi)}{\sqrt{2} (\sigma_\xi^2 + \sigma_\varphi^2 - 2\rho \sigma_\xi \sigma_\varphi)} \right) \right) + \\
&+ \bar{c}^2 \exp (\sigma_\xi^2) \left( 1 + \text{erf} \left( \frac{\bar{y} - \bar{x} + 2\sigma_\xi (\rho \sigma_\varphi - \sigma_\xi)}{\sqrt{2} (\sigma_\xi^2 + \sigma_\varphi^2 - 2\rho \sigma_\xi \sigma_\varphi)} \right) \right) + \\
&- \bar{p} \bar{c} \exp (\rho \sigma_\xi \sigma_\varphi) \left( 1 + \text{erf} \left( \frac{\bar{y} - \bar{x} + \sigma_\varphi^2 - \sigma_\xi^2}{\sqrt{2} (\sigma_\xi^2 + \sigma_\varphi^2 - 2\rho \sigma_\xi \sigma_\varphi)} \right) \right) - (E[\max (p - c, 0)])^2 \\
\text{Cov}[\max(p - c, 0), c] &= \int_{\mathbb{R}^2} \max (e^y - e^x, 0) e^x P(x, y) \, dx \, dy - E[\max (p - c, 0)] E[c] =
\end{align*}
\]
\[ E \left[ \max (p, c) \right] = E \left[ \max (p - c, 0) \right] + E [c] = \]

\[ = \frac{1}{2} \left( 1 + \text{erf} \left( \frac{\bar{y} - \bar{x} + \sigma_\phi (\sigma_\phi - \rho \sigma_\xi)}{\sqrt{2 (\sigma_\xi^2 + \sigma_\phi^2 - 2 \rho \sigma_\xi \sigma_\phi)}} \right) \right) \bar{p} + \]

\[ + \frac{1}{2} \left( 1 - \text{erf} \left( \frac{\bar{y} - \bar{x} + \sigma_\xi (\rho \sigma_\phi - \sigma_\xi)}{\sqrt{2 (\sigma_\xi^2 + \sigma_\phi^2 - 2 \rho \sigma_\xi \sigma_\phi)}} \right) \right) \bar{c} \]

\[ V \left[ \max (p, c) \right] = V \left[ \max (p - c, 0) \right] + V [c] + 2 \text{Cov} \left[ \max (p - c, 0), c \right] \]

Note that

\[ \text{Cov} [p, c] = E [p] E [c] \exp (\rho \sigma_\xi \sigma_\phi) \]

\[ \text{Corr} [p, c] = \frac{\exp (\rho \sigma_\xi \sigma_\phi)}{\sqrt{\left( \exp (\sigma_\xi^2) - 1 \right) \left( \exp (\sigma_\phi^2) - 1 \right)}} \]

so that \( \text{Corr} [p, c] \neq \rho \).